

ch4.4 p1
ex

MA121

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Given that the potential number of customers is
 $\Rightarrow L = 5500$

Also, 500 customers will purchase as soon as
the product is released,

$$\Rightarrow S(0) = 500$$

logistic
growth

$\Rightarrow$ ~~Relative difference~~ between potential number
of customers and current number of customers
shrinks by 4%

$$\Rightarrow \frac{L-S}{S} = 4\% = 0.04$$

using the Logistic Growth model,

$$S(t) = \frac{L}{1 + ae^{-kt}}$$

$$i) L = 5500, k = 0.04$$

In a logistic growth model, $S(0) = \frac{L}{1+a}$

We are told that $S(0) = 500$

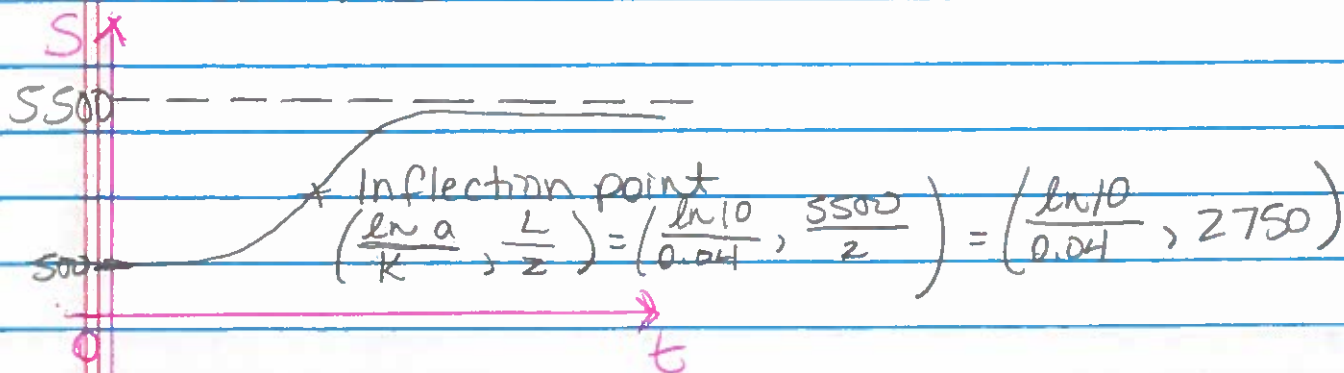
$$\Rightarrow \frac{L}{1+a} = 500$$

$$\Rightarrow \frac{5500}{1+a} = 500$$

$$\Rightarrow 1+a = \frac{5500}{500} = 11$$

$$\Rightarrow a = 11 - 1 = 10$$

$$\therefore S(t) = \frac{5500}{1 + 10e^{-0.04t}}$$



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ii) At $t = 60$,

$$S(60) = \frac{5500}{1 + 10e^{-0.04(60)}}$$

$$= \frac{5500}{1 + 10e^{-2.4}}$$

$$= 2,883.839\dots$$

 ≈ 2884 cups of coffee.iii) When $S = 2750$,

$$2750 = \frac{5500}{1 + 10e^{-0.04t}}$$

$$\Rightarrow 1 + 10e^{-0.04t} = \frac{5500}{2750} = 2$$

$$\Rightarrow 10e^{-0.04t} = 2 - 1 = 1$$

$$\Rightarrow e^{-0.04t} = \frac{1}{10} = 0.1$$

$$\Rightarrow e^{-0.04t} = 0.1$$

$$\Rightarrow \log_e 0.1 = -0.04t$$

$$\Rightarrow \ln 0.1 = 0.04t$$

$$\Rightarrow t = \frac{\ln 0.1}{-0.04}$$

$$\Rightarrow t \approx 57.5646\dots$$

 $\therefore 2750$ cups will be sold on the 58th day.

iv) The point in time where sales increase at a slower rate happens at the inflection point. ie at $t = \frac{\ln 10}{0.04} = 57.5646$, or the 58th day.

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Given Capacity is 5000 when $t=0$, $S(0)=20$ When $t=10$, $S(10)=1200$

i) learning curve model

$$S(t) = L - ae^{-kt}$$

$$L = 5000 \text{ when } t=0, S(0) = L - a = 20$$

$$\Rightarrow 5000 - a = 20$$

$$\Rightarrow 5000 - 20 = a$$

$$\Rightarrow a = 4980$$

$$\text{when } t=10, S(10) = 1200$$

$$\Rightarrow L - a^{-k(10)} = 1200$$

$$\Rightarrow 5000 - 4980e^{-10k} = 1200, \text{ since } L=5000 \text{ and } a=4980$$

$$\Rightarrow 4980e^{-10k} = 5000 - 1200$$

$$= 3800$$

$$\Rightarrow e^{-10k} = \frac{3800}{4980}$$

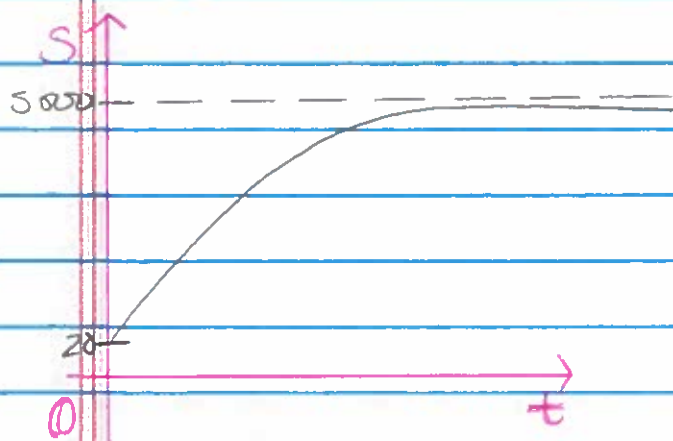
$$\Rightarrow \log_e \frac{3800}{4980} = -10k$$

$$\Rightarrow k = \frac{1}{10} \ln \frac{3800}{4980}$$

$$\Rightarrow 0.027042882 \dots$$

$$\approx 0.027043$$

$$\therefore S(t) = 5000 - 4980e^{-0.027043t}$$

ii) when $t=20$,

$$S(20) = 5000 - 4980e^{-0.027043(20)}$$

$$S(20) = 5000 - 4980e^{-0.54086}$$

$$S(20) = 2100.408425$$

$$\approx 2100 \text{ tickets.}$$

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(ii) When $S = 4500$,

$$4500 = 5000 - 4980 e^{-0.027043t}$$

$$\Rightarrow 4980 e^{-0.027043t} = 5000 - 4500 = 500$$

$$\Rightarrow e^{-0.027043t} = \frac{500}{4980} = \frac{50}{498}$$

$$\Rightarrow \log_e \frac{50}{498} = -0.027043t$$

$$\Rightarrow t = \frac{1}{0.027043} \ln \frac{50}{498}$$

$$\approx 84.99 \dots$$

$\therefore S = 4500$ tickets will be sold on the 85th day.

4.5

Derivatives of Exponential and Logarithmic Functions

The following functions have been run through the derivative formula.

$\hookrightarrow$ given $y = f(x)$

$$\Rightarrow \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

i) The derivative of the exponential form.

* $y = b^{f(x)}$; where $b > 0$
 $\Rightarrow \frac{dy}{dx} = b^{f(x)} \cdot \ln b \cdot f'(x)$

ii) The derivative of the logarithmic function

$y = \log_b g(x)$; where $0 < b < 1$ (or) $b > 1$ (AND) $g(x) > 0$
 $\Rightarrow \frac{dy}{dx} = \frac{g'(x)}{g(x) \ln b}$

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$$y = e^{-5x} + e^{5x}$$

$$\Rightarrow \frac{dy}{dx} = e^{-5x} \cdot \ln e \cdot (-5x)' + e^{5x} \cdot \ln e \cdot (5x)'$$

$$\Rightarrow e^{-5x} (1)(-5) + e^{5x} (1)(5)$$

$$\Rightarrow \frac{dy}{dx} = -5e^{-5x} + 5e^{5x}$$

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Special case.

$$y = e^{f(x)}$$

$$\Rightarrow \frac{dy}{dx} = e^{f(x)} \cdot \ln e \cdot f'(x)$$

$$\Rightarrow \frac{dy}{dx} = e^{f(x)} f'(x)$$

Recall that $\log_b b = 1$; $\log_e e = \ln e = 1$

Given $y = e^x$

$$\Rightarrow \frac{dy}{dx} = e^x \ln e \cdot (x)'$$

$$\Rightarrow \frac{dy}{dx} = e^x (1)(1) = e^x$$

So when $y = e^x$

$$\Rightarrow \frac{dy}{dx} = e^x$$

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$$y = 10^{x^2+11} - 4^x$$

$$\Rightarrow \frac{dy}{dx} = 10^{x^2+11} \cdot \ln 10 \cdot (x^2+11)' - 4^x \cdot \ln 4 \cdot (x)'$$

$$= 10^{x^2+11} \cdot \ln 10 \cdot (2x) - 4^x \ln 4$$

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$$y = \ln(8 + \sqrt{x} - x^3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(8 + \sqrt{x} - x^3)'}{(8 + \sqrt{x} - x^3) \ln e}$$

$$= \frac{1}{2\sqrt{x}} - 3x^2$$

$$(8 + \sqrt{x} - x^3)(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}} - 3x^2}{8 + \sqrt{x} - x^3}$$

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Special case

given $y = \ln g(x)$
 $\Rightarrow \frac{dy}{dx} = \frac{g'(x)}{g(x) \ln e}$ since $\ln e = \log_e e = 1$

$y = \ln g(x)$
 $\Rightarrow \frac{dy}{dx} = \frac{g'(x)}{g(x)}$

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$y = 10 \log_{10} (x^4 - 2x)$
 $\Rightarrow \frac{dy}{dx} = 10 \frac{(x^4 - 2x)'}{(x^4 - 2x) \ln 10}$
 $= \frac{10(4x^3 - 2)}{(x^4 - 2x) \ln 10}$

Applications of Logarithmic Differentiation

↳ complicated functions

~~Step one:~~ Take the natural logarithm of both sides.

~~Step two:~~ Simplify as much as possible using the Laws of Logarithms (refer to p 113)

(ex i) $\log_b (gh) = \log_b g + \log_b h$

(ex ii) $\log_b \frac{g}{h} = \log_b g - \log_b h$

(ex iii) $\log_b g^s = s \log_b g$

$g \log_b (gh)^s$

$= s \log_b (gh)$

$= s(\log_b g + \log_b h)$

~~Step three:~~ Implicitly differentiate both sides

~~Step four:~~ Simplify as much as possible

~~Step five:~~ rewrite $\frac{dx}{dy}$ in terms of both x and y .

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$$y = (x^2 + 1)^3 (3 - x)$$

$$\Rightarrow \ln y = \ln [(x^2 + 1)^3 (3 - x)]$$

$$= \ln (x^2 + 1)^3 + \ln (3 - x)$$

$$\Rightarrow \ln y = 3 \ln (x^2 + 1) + \ln (3 - x)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 3 \frac{(x^2 + 1)'}{x^2 + 1} + \frac{(3 - x)'}{3 - x}$$

$$= 3 \frac{2x}{x^2 + 1} + \frac{-1}{3 - x}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{6x}{x^2 + 1} - \frac{1}{3 - x} \right)$$

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$$y = \left(\frac{e^x}{x+1} \right)^2$$

$$\Rightarrow \ln y = \ln \left(\frac{e^x}{x+1} \right)^2$$

$$= 2 \ln \frac{e^x}{x+1}$$

$$= 2 (\ln e^x - \ln (x+1))$$

$$= 2 (x - \ln (x+1))$$

$$\text{Recall } \log_b b^x = x$$

$$\text{So, } \ln e^x = \log_e e^x = x$$

$$\text{From } \ln y = 2 (x - \ln (x+1))$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2 \left(1 - \frac{(x+1)'}{x+1} \right)$$

$$= 2 \left(1 - \frac{1}{x+1} \right)$$

$$\Rightarrow \frac{dy}{dx} = 2y \left(1 - \frac{1}{x+1} \right)$$

$$= 2y \left(\frac{x+1}{x+1} - \frac{1}{x+1} \right) = 2y \left(\frac{x}{x+1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x+1}$$

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Important!

When $y = f(x) = [f(x)]^{g(x)}$

we must use logarithmic differentiation to find its derivative.

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$$y = x^{2x^2 - 5x - 1}$$

$$\Rightarrow \ln y = \ln x^{2x^2 - 5x - 1}$$

$$= (2x^2 - 5x - 1) \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\ln x)(2x^2 - 5x - 1)' + (2x^2 - 5x - 1)(\ln x)'$$

$$= (\ln x)(4x - 5) + (2x^2 - 5x - 1)\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = y \left((4x - 5) \ln x + \frac{2x^2 - 5x - 1}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = y \left((4x - 5) \ln x + 2x - 5 - \frac{1}{x} \right)$$

Notice that $y = \ln x$

$$\Rightarrow \frac{dy}{dx} = \frac{(x)'}{x \ln x} = \frac{1}{(x)(1)} = \frac{1}{x}$$

If $y = \ln x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

Q Find the equation of the tangent line to $y = x^x$ at $x = 1$
we need $\frac{dy}{dx}$, the slope function, given $y = x^x \Rightarrow \ln y = \ln x^x$

$$\Rightarrow \ln y = x \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\ln x)(x)' + (x)(\ln x)'$$

$$= (\ln x)(1) + (x)\left(\frac{1}{x}\right)$$

$$= \ln x + 1$$

$$\Rightarrow \frac{dy}{dx} = y(\ln x + 1)$$

At $x = 1$, $y = 1' = 1$

So $m = \frac{dy}{dx} = (1)(\ln 1 + 1)$

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$$\Rightarrow m = (1)(0+1) = 1$$

And we can use the point $(x_1, y_1) = (1, 1)$
so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 1 = 1(x - 1) = x - 1$$

$$\Rightarrow y = x$$